ISRAEL JOURNAL OF MATHEMATICS 117 (2000), 93–104

ON TOTALLY UMBILICAL SUBMANIFOLDS OF S^{n+p}

BY

HUAFEI SUN*

Department of Mathematics, Kumamoto University, Kumamoto 860-8555, Japan
e-mail: sun@math.sci.kumamoto-u.ac.jp

ABSTRACT

In this paper, we will give some conditions under which the submanifolds in a unit sphere are totally umbilical.

0. Introduction

Let S^{n+p} be an $(n+p)$-dimensional unit sphere of constant curvature 1 and M^n an n-dimensional compact submanifold isometrically immersed in S^{n+p}. The following results are well-known.

THEOREM A [4]: *Let M^n be an n-dimensional compact minimal submanifold in S^{n+p}, $p > 1$. If the scalar curvature of M^n is larger than or equal to $\frac{1}{3}n(3n-5)$, then M^n is totally geodesic or a Veronese surface in S^4.*

THEOREM B [9]: *Let M^n be an n-dimensional compact minimal submanifold in S^{n+p}. If the sectional curvature of M^n is larger than or equal to $(p-1)/(2p-1)$, then M^n is totally geodesic, the standard immersion of the product of the two spheres or a Veronese surface in S^4.*

THEOREM C [3]: *Let M^n be an n-dimensional compact minimal submanifold in S^{n+p}, $n > 4$. If the Ricci curvature of M^n is larger than $n-2$, then M^n is totally geodesic.*

Let h be the second fundamental form of the immersion and ξ be the mean curvature vector; $< \cdot, \cdot >$ denotes the scalar product of S^{n+p}. If there exists a function λ on M^n such that

$$(*) \qquad < h(X,Y), \xi > = \lambda < X, Y >$$

* This research was partially supported by JSPS.
Received June 23, 1997

for any tangent vectors X, Y on M^n, then M^n is called a pseudo-umbilical submanifold of S^{n+p} (cf. [1]). It is clear that $\lambda \geq 0$, and it is known that the mean curvature $H = |\xi|$ of M^n is constant (cf. Lemma 1). If the mean curvature vector $\xi = 0$ identically, then M^n is called a minimal submanifold of S^{n+p}. Every minimal submanifold of S^{n+p} is itself a pseudo-umbilical submanifold.

In this paper, we study pseudo-umbilical submanifolds and generalize Theorems A, B and C. We assume that the mean curvature H of M^n is not zero. Our results are the following:

THEOREM 1: *Let M^n be an n-dimensional compact pseudo-umbilical submanifold in S^{n+p}, $p > 1$. If the scalar curvature of M^n is larger than or equal to $\frac{1}{3}n(3n-5)(1+H^2)$, then M^n is totally umbilical or $n = 2$ and M^2 is a Veronese surface in $S^4(\frac{1}{\sqrt{1+H^2}})$.*

THEOREM 2: *Let M^n be an n-dimensional compact pseudo-umbilical submanifold in S^{n+p}, $p > 1$. If the sectional curvature of M^n is larger than or equal to $\frac{3p-5}{6(p-1)}(1+H^2)$, then M^n is totally umbilical or $n = 2$ and M^2 is a Veronese surface in $S^4(\frac{1}{\sqrt{1+H^2}})$.*

THEOREM 3: *Let M^n be an n-dimensional compact pseudo-umbilical submanifold in S^{n+p}, $p > 1$, $n > 4$. If the Ricci curvature of M^n is larger than $(n-2)(1+H^2)$, then M^n is totally umbilical.*

Remark: It is clear that our results generalize Theorem A, B and C.

ACKNOWLEDGEMENT: I would like to thank K. Ogiue for his advice and encouragement and would like to express my thanks to the referee for valuable suggestions.

1. Local formulas

Let S^{n+p} be an $(n+p)$-dimensional unit sphere of constant curvature 1 and M^n an n-dimensional pseudo-umbilical manifold isometrically immersed in S^{n+p}. We choose a local field of orthonormal frames $e_1, \ldots, e_{n+p}$ in S^{n+p} such that $e_1, \ldots, e_n$ are tangent to M^n. We make use of the following convention on the ranges of indices:

$$A, B, \ldots = 1, \ldots, n+p; \quad i, j, \ldots = 1, \ldots, n; \quad \alpha, \beta, \ldots = n+1, \ldots, n+p.$$

Then the structure equations of S^{n+p} are given by

$$d\omega_A = -\sum_B \omega_{AB} \wedge \omega_B, \omega_{AB} + \omega_{BA} = 0,$$
$$d\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} + \frac{1}{2}\sum_{CD} K_{ABCD}\omega_C \wedge \omega_D,$$
$$K_{ABCD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}.$$

Restrict these forms to M^n. Then

$$\omega_\alpha = 0, \quad \omega_{i\alpha} = \sum_j h^\alpha_{ij}\omega_j, \quad h^\alpha_{ij} = h^\alpha_{ji},$$
$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j,$$
$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2}\sum_{kl} R_{ijkl}\omega_k \wedge \omega_l,$$
$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}), \tag{1.1}$$

$$d\omega_\alpha = -\sum_\beta \omega_{\alpha\beta} \wedge \omega_\beta,$$
$$d\omega_{\alpha\beta} = -\sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \frac{1}{2}\sum_{ij} R_{\alpha\beta ij}\omega_i \wedge \omega_j,$$
$$R_{\alpha\beta ij} = \sum_k (h^\alpha_{ki}h^\beta_{kj} - h^\alpha_{kj}h^\beta_{ki}). \tag{1.2}$$

We call $H = |\xi| = \frac{1}{n}\sqrt{\sum_\alpha(\sum_i h^\alpha_{ii})^2}$ the mean curvature of M^n and $S = \sum_{ij\alpha}(h^\alpha_{ij})^2$ the square of the length of h; h^α_{ijk} and h^α_{ijkl} are defined by

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} + \sum_k h^\alpha_{ik}\omega_{kj} + \sum_k h^\alpha_{kj}\omega_{ki} - \sum_\beta h^\beta_{ij}\omega_{\beta\alpha} \tag{1.3}$$

and

$$\sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} + \sum_l h^\alpha_{ijl}\omega_{lk} + \sum_l h^\alpha_{ilk}\omega_{lj} + \sum_l h^\alpha_{ljk}\omega_{li} - \sum_\beta h^\beta_{ijk}\omega_{\beta\alpha}$$

respectively;

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{im}R_{mjkl} + \sum_m h^\alpha_{mj}R_{mikl} + \sum_\beta h^\beta_{ij}R_{\beta\alpha kl}$$

where $h^\alpha_{ijk} = h^\alpha_{ikj}$. By (1.1) we have

$$R_{ij} = (n-1)\delta_{ij} + \sum_\alpha (h^\alpha_{ij} \sum_k h^\alpha_{kk}) - \sum_{k\alpha} h^\alpha_{ki} h^\alpha_{kj}. \tag{1.4}$$

Now, let ξ be parallel to e_{n+p}; then

$$\operatorname{tr} H_{n+p} = nH, \quad \operatorname{tr} H_\alpha = 0, \quad \alpha \neq n+p. \tag{1.5}$$

In order to prove our Theorems we need the following:

LEMMA 1 [5]: *The mean curvature of pseudo-umbilical submanifolds in a space form of constant curvature is constant.*

LEMMA 2 [4]: *Let H_i $(i \geq 2)$ be symmetric $(n \times n)$-matrices, $S_i = \operatorname{tr} H_i^2$ and $S = \sum_i S_i$. Then*

$$\sum_{ij} N(H_iH_j - H_jH_i) + \sum_{ij} (\operatorname{tr} H_iH_j)^2 \leq \frac{3}{2} S^2$$

and equality holds if and only if all $H_i = 0$ or there exist two H_i different from zero. Moreover, if $H_1 \neq 0$, $H_2 \neq 0$, $H_i = 0$ $(i \neq 1, 2)$, then $S_1 = S_2$ and there exists an orthogonal $(n \times n)$-matrix T such that

$$TH_1{}^tT = \begin{pmatrix} f & 0 & 0 \\ 0 & -f & \\ 0 & & 0 \end{pmatrix}, \quad TH_2{}^tT = \begin{pmatrix} 0 & f & 0 \\ f & 0 & \\ 0 & & 0 \end{pmatrix}, \quad \text{where } f = \sqrt{\frac{S_1}{2}}.$$

Using Lemma 1 and a direct calculation we have (cf. [2, 9])

$$\begin{aligned} \frac{1}{2}\Delta \sum_{ij\alpha\neq n+p} (h^\alpha_{ij})^2 &= \sum_{ijk\alpha\neq n+p} (h^\alpha_{ijk})^2 + \sum_{ij\alpha\neq n+p} h^\alpha_{ij} \Delta h^\alpha_{ij} \\ &= \sum_{ijk\alpha\neq n+p} (h^\alpha_{ijk})^2 + \sum_{ijk\alpha\neq n+p} h^\alpha_{ij} h^\alpha_{kkij} \\ &\quad + \sum_{ijkl\alpha\neq n+p} h^\alpha_{ij} h^\alpha_{lk} R_{lijk} + \sum_{ijkl\alpha\neq n+p} h^\alpha_{ij} h^\alpha_{li} R_{lkik} \\ &\quad + \sum_{ijk\alpha\neq n+p\beta} h^\alpha_{ij} h^\beta_{ki} R_{\alpha\beta kj}. \end{aligned} \tag{1.6}$$

2. Proofs of Theorems

From (*) and (1.5) we get $\sum_\alpha \operatorname{tr} H_\alpha h_{ij}^\alpha = n\lambda\delta_{ij}$, $H^2 = \lambda$ and

$$h_{ij}^{n+p} = H\delta_{ij}. \tag{2.1}$$

Using (2.1) and (1.4) we obtain

$$\begin{aligned}
(2.2)\ \frac{1}{2}\Delta \sum_{ij\alpha\neq n+p} (h_{ij}^\alpha)^2 \\
= & \sum_{ijk\alpha\neq n+p} (h_{ijk}^\alpha)^2 + n \sum_{ij\alpha\neq n+p} (h_{ij}^\alpha)^2 + \sum_{ijmk\beta\alpha\neq n+p} h_{ij}^\alpha h_{mi}^\beta h_{jm}^\alpha h_{kk}^\beta \\
& + 2\Bigg[\sum_{\beta\alpha\neq n+p} \operatorname{tr}(H_\alpha H_\beta)^2 - \sum_{\beta\alpha\neq n+p} \operatorname{tr}(H_\alpha^2 H_\beta^2)\Bigg] - \sum_{\beta\alpha\neq n+p} [\operatorname{tr}(H_\alpha H_\beta)]^2 \\
= & \sum_{ijk\alpha\neq n+p} (h_{ijk}^\alpha)^2 + n \sum_{ij\alpha\neq n+p} (h_{ij}^\alpha)^2 + \sum_{ijkm\beta\alpha\neq n+p} h_{ij}^\alpha h_{mi}^\alpha h_{mj}^\beta h_{kk}^\beta \\
& + 2\Bigg[\sum_{\beta,\alpha\neq n+p} \operatorname{tr}(H_\alpha H_\beta)^2 - \sum_{\beta,\alpha\neq n+p} \operatorname{tr}(H_\alpha^2 H_\beta^2)\Bigg] - \sum_{\beta,\alpha\neq n+p} [\operatorname{tr}(H_\alpha H_\beta)]^2 \\
& + 2\Bigg[\sum_{\alpha\neq n+p} \operatorname{tr}(H_\alpha H_{n+p})^2 - \sum_{\alpha\neq n+p} \operatorname{tr}(H_\alpha^2 H_{n+p}^2)\Bigg] - \sum_{\alpha\neq n+p} [\operatorname{tr}(H_\alpha H_{n+p})]^2 \\
= & \sum_{ijk\alpha\neq n+p} (h_{ijk}^\alpha)^2 + n \sum_{ij\alpha\neq n+p} (h_{ij}^\alpha)^2 + nH^2 \sum_{\alpha\neq n+p} (h_{ij}^\alpha)^2 \\
& - \sum_{\beta,\alpha\neq n+p} N(H_\alpha H_\beta - H_\beta H_\alpha) - \sum_{\beta,\alpha\neq n+p} [\operatorname{tr}(H_\alpha H_\beta)]^2.
\end{aligned}$$

When $p = 2$, by a simple calculation from (2.2) we get

$$\frac{1}{2}\Delta \sum_{ij} (h_{ij}^{n+1})^2 \geq [n(1+2H^2) - S] \sum_{ij} (h_{ij}^{n+1})^2.$$

It shows that when $S < n(1+2H^2)$, then $\sum_{ij}(h_{ij}^{n+1})^2 = 0$ and M^n is totally umbilical.

When $p \geq 3$, applying Lemma 2 to (2.2) we get

(2.3)
$$\begin{aligned}
&\frac{1}{2}\Delta \sum_{ij\alpha\neq n+p} (h_{ij}^{\alpha})^2 \\
&\geq \sum_{ijk\alpha\neq n+p} (h_{ijk}^{\alpha})^2 + n \sum_{ij\alpha\neq n+p} (h_{ij}^{\alpha})^2 - \frac{3}{2}\Big[\sum_{\alpha\neq n+p} \operatorname{tr} H_{\alpha}^2\Big]^2 + nH^2 \sum_{ij\alpha\neq n+p} (h_{ij}^{\alpha})^2 \\
&\geq n \sum_{ij\alpha\neq n+p} (h_{ij}^{\alpha})^2 - \frac{3}{2} \sum_{\alpha\neq n+p} (h_{ij}^{\alpha})^2 (S - nH^2) + nH^2 \sum_{ij\alpha\neq n+p} (h_{ij}^{\alpha})^2 \\
&= \Big(n - \frac{3}{2}S + \frac{5}{2}nH^2\Big) \sum_{ij\alpha\neq n+p} (h_{ij}^{\alpha})^2.
\end{aligned}$$

Since M^n is compact, we see from (2.2) that when

$$S \leq \frac{2}{3}n\Big(1 + \frac{5}{2}H^2\Big)$$

or the scalar curvature R of M^n satisfies

$$R \geq \frac{1}{3}n(3n-5)(1+H^2),$$

then $\sum_{ij\alpha\neq n+p}(h_{ij}^{\alpha})^2 = 0$, i.e. M^n is totally umbilical or $S = \frac{2}{3}n(1+\frac{5}{2}H^2)$. In the latter case, using the same method as in [2] we conclude that $n = 2$ and the equality

$$\sum_{\alpha,\beta\neq n+p} N(H_\alpha H_\beta - H_\beta H_\alpha) + \sum_{\alpha,\beta\neq n+p} (\operatorname{tr} H_\alpha H_\beta)^2 = \frac{3}{2}\Big[\sum_{ij\alpha\neq n+p} (h_{ij}^{\alpha})^2\Big]^2$$

holds. Thus we may assume

(2.4)
$$H_{n+1} = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad H_{n+2} = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, \quad H_{n+p} = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}, \quad H_\alpha = 0,$$

where $a \neq 0$, $\alpha \neq n+1, n+2, n+p$.

Now we put

$$S_\alpha = \sum_{ij} (h_{ij}^{\alpha})^2,$$
$$p\sigma_1 = \sum_{\alpha} S_\alpha = S,$$
$$p(p-1)\sigma_2 = 2\sum_{\alpha<\beta} S_\alpha S_\beta.$$

It can be seen easily (cf. [2]) that

$$p^2(p-1)(\sigma_1^2-\sigma_2)=\sum_{\alpha<\beta}(S_\alpha-S_\beta)^2. \tag{2.5}$$

By a direct calculation, using (2.4), we get

$$p^2(p-1)\sigma_1^2=(p-1)(4a^2+2H^2)^2, \tag{2.6}$$

$$p^2(p-1)\sigma_2=p(8a^4+16a^2H^2), \tag{2.7}$$

and

$$\sum_{\alpha<\beta}(S_\alpha-S_\beta)^2=8(a^2-H^2)^2. \tag{2.8}$$

Substituting (2.6) – (2.8) into (2.5) we obtain

$$(p-1)(4a^2+2H^2)^2-p(8a^4+16a^2H^2)=8(a^2-H^2)^2. \tag{2.9}$$

From (2.9) we conclude

$$(p-3)(2a^4+H^4)=0,$$

because $2a^4+H^4\neq 0$, which implies $p=3$. Thus by [2], we know that M^2 is a Veronese surface in $S^4(\frac{1}{\sqrt{1+H^2}})$. Since $\frac{2}{3}n(1+\frac{5}{2}H^2)<n(1+2H^2)$, when $p=2$ and $S\leq\frac{3}{2}n(1+\frac{5}{2}H^2)$ or $R\geq\frac{(3n-5)}{n}(1+H^2)$, then M^n is totally umbilical. This completes the proof of Theorem 1.

Proof of Theorem 2: For any positive real number a $(0<a<1)$, we have

$$\begin{aligned}
\frac{1}{2}\Delta\sum_{ij\alpha\neq n+p}(h_{ij}^\alpha)^2
&=\sum_{ijk\alpha\neq n+p}(h_{ijk}^\alpha)^2+(1+a)\Big(\sum_{ijkl\alpha\neq n+p}h_{ij}^\alpha h_{lk}^\alpha R_{lijk}+\sum_{ijkl\alpha\neq n+p}h_{ij}^\alpha h_{li}^\alpha R_{lkik}\Big)\\
&\quad+\sum_{ijk\alpha\neq n+p\beta}h_{ij}^\alpha h_{ki}^\beta R_{\alpha\beta kj}-a\Big(\sum_{ijkl\alpha\neq n+p}h_{ij}^\alpha h_{lk}^\alpha R_{lijk}+\sum_{ijkl\alpha\neq n+p}h_{ij}^\alpha h_{li}^\alpha R_{lkik}\Big)\\
&=\sum_{ijk\alpha\neq n+p}(h_{ijk}^\alpha)^2+(1+a)\Big(\sum_{ijkl\alpha\neq n+p}h_{ij}^\alpha h_{lk}^\alpha R_{lijk}+\sum_{ijkl\alpha\neq n+p}h_{ij}^\alpha h_{li}^\alpha R_{lkik}\Big)\\
&\quad-na\sum_{ij\alpha\neq n+p}(h_{ij}^\alpha)^2-nH^2a\sum_{ij\alpha\neq n+p}(h_{ij}^\alpha)^2\\
&\quad+\frac{1}{2}(a-1)\sum_{\beta\alpha\neq n+p}N(H_\alpha H_\beta-H_\beta H_\alpha)+a\sum_{\alpha\neq n+p}(\operatorname{tr}H_\alpha H_\beta)^2.
\end{aligned} \tag{2.10}$$

For fixed $\alpha \neq n+p$, we can choose $e_1, \dots, e_n$ such that $h^\alpha_{ij} = h^\alpha_{ii}\delta_{ij}$. Thus, we have (cf. [9])

$$\sum_{ijkl\alpha\neq n+p} h^\alpha_{ij}h^\alpha_{lk}R_{lijk} + \sum_{ijkl\alpha\neq n+p} h^\alpha_{ij}h^\alpha_{li}R_{lkik} \geq nK \sum_{ij\alpha\neq n+p} (h^\alpha_{ij})^2, \tag{2.11}$$

where K is the infimum of the sectional curvature of M^n.

When $p=2$, from (2.10) and (2.11) we get

$$\frac{1}{2}\Delta\sum_{ij}(h^{n+1}_{ij})^2 \geq (a+1)nK\sum_{ij}(h^{n+1}_{ij})^2 - na(1+H^2)\sum_{ij}(h^{n+1}_{ij})^2 + a[\sum_{ij}(h^{n+1}_{ij})^2]^2.$$

It shows that when

$$K \geq \frac{a}{a+1}(1+H^2),$$

then $\sum_{ij}(h^{n+1}_{ij})^2 = 0$ and M^n is totally umbilical.

When $p \geq 3$, combining Lemma 2, (2.11) with (2.10) we get

$$\begin{aligned} \frac{1}{2}\Delta \sum_{ij\alpha\neq n+p} (h^\alpha_{ij})^2 \geq & [(1+a)nK - na - nH^2 a] \sum_{ij\alpha\neq n+p} (h^\alpha_{ij})^2 \\ & - \frac{3}{4}(1-a)(\sum_{\alpha\neq n+p} \operatorname{tr} H^2_\alpha)^2 \\ & + \frac{1}{2}(1+a) \sum_{\alpha,\beta\neq n+p} (\operatorname{tr} H_\alpha H_\beta)^2. \end{aligned} \tag{2.12}$$

On the other hand, we can choose $e_{n+1}, \dots, e_{n+p-1}$ such that $\operatorname{tr} H_\alpha H_\beta = \operatorname{tr} H^2_\alpha \delta_{\alpha\beta}$. So we have

$$\sum_{\alpha,\beta\neq n+p} (\operatorname{tr} H_\alpha H_\beta)^2 = \sum_{\alpha\neq n+p} (\operatorname{tr} H^2_\alpha)^2. \tag{2.13}$$

The following inequality is obvious,

$$\sum_{\alpha\neq n+p} (\operatorname{tr} H^2_\alpha)^2 \geq \frac{1}{p-1}\left(\sum_{\alpha\neq n+p} \operatorname{tr} H^2_\alpha\right)^2, \tag{2.14}$$

and the equality in (2.14) holds if and only if all $\operatorname{tr} H^2_\alpha$ are equal. Substituting (2.13) and (2.14) into (2.12) and taking $a = (3p-5)/(3p-1)$, we get

(2.15)

$$\frac{1}{2}\Delta \sum_{ij\alpha\neq n+p} (h^\alpha_{ij})^2 \geq \left[\frac{6(p-1)n}{3p-1}K - \frac{(3p-5)n}{3p-1} - \frac{(3p-5)n}{3p-1}H^2\right] \sum_{ij\alpha\neq n+p} (h^\alpha_{ij})^2.$$

So, when

$$K \geq \frac{3p-5}{6(p-1)}(1+H^2), \tag{2.16}$$

then $\sum_{ij\alpha\neq n+p}(h_{ij}^{\alpha})^2 = 0$, i.e. M^n is totally umbilical or

$$K = \frac{3p-5}{6(p-1)}(1+H^2).$$

In the latter case, the equality

$$\sum_{\alpha,\beta\neq n+p} N(H_\alpha H_\beta - H_\beta H_\alpha) + \sum_{\alpha,\beta\neq n+p} (\operatorname{tr} H_\alpha H_\beta)^2 = \frac{3}{2}\Big[\sum_{ij\alpha\neq n+p}(h_{ij}^{\alpha})^2\Big]^2$$

holds. Thus by Lemma 2 we see that all $H_\alpha = 0$, $\alpha \neq n+p$ and M^n is totally umbilical or there exist only two of $H_\alpha \neq 0$, $\alpha \neq n+p$. However, the equality (2.16) implies equality (2.14), so that all $\operatorname{tr} H_\alpha^2$ are equal. This is a contradiction. This proves Theorem 2. ∎

Proof of Theorem 3: We compute directly from (1.6):

$$\begin{aligned}
&\frac{1}{2}\Delta \sum_{ij\alpha\neq n+p}(h_{ij}^{\alpha})^2 \\
&= \sum_{ijk\alpha\neq n+p}(h_{ijk}^{\alpha})^2 + n(1+H^2)\sum_{ij\alpha\neq n+p}(h_{ij}^{\alpha})^2 \\
&\quad - \sum_{\alpha,\beta\neq n+p} N(H_\alpha H_\beta - H_\beta H_\alpha) - \sum_{\alpha,\beta\neq n+p}(\operatorname{tr} H_\alpha H_\beta)^2 \\
&= \sum_{ijk\alpha\neq n+p}(h_{ijk}^{\alpha})^2 + n(1+H^2)\sum_{ij\alpha\neq n+p}(h_{ij}^{\alpha})^2 - \sum_{\alpha,\beta\neq n+p} N(H_\alpha H_\beta - H_\beta H_\alpha) \\
&\quad - \sum_{\alpha\neq n+p}(\operatorname{tr} H_\alpha^2)^2.
\end{aligned} \tag{2.17}$$

When $p = 2$, from (2.17) we get

$$\frac{1}{2}\Delta\sum_{ij}(h_{ij}^{n+1})^2 \geq n(1+H^2)\sum_{ij}(h_{ij}^{n+1})^2 - \Big(\sum_{ij}(h_{ij}^{n+1})\Big)^2.$$

On the other hand, by the assumption in Theorem 3 we get

$$(n-2)(1+H^2) < R_{ii} = (n-1)(1+H^2) - \sum(h_{ij}^{n+1})^2$$

and so

$$\sum_{ij}(h_{ij}^{n+1})^2 < n(1+H^2).$$

Combining these two inequalities we get

$$\begin{aligned}\frac{1}{2}\Delta\sum_{ij}(h_{ij}^{n+1})^2 \geq& n(1+H^2)\sum_{ij}(h_{ij}^{n+1})^2 - \Big(\sum_{ij}(h_{ij}^{n+1})^2\Big)^2\\ \geq& n(1+H^2)\sum_{ij}(h_{ij}^{n+1})^2 - n(1+H^2)\sum_{ij}(h_{ij}^{n+1})^2 = 0.\end{aligned}$$

From this we see that $\sum_{ij}(h_{ij}^{n+1})^2$ is constant and

$$\Big[n(1+H^2) - \sum_{ij}(h_{ij}^{n+1})^2\Big]\sum_{ij}(h_{ij}^{n+1})^2 = 0,$$

which implies that $\sum_{ij}(h_{ij}^{n+1})^2 = 0$, i.e. M^n is totally umbilical since $\sum_{ij}(h_{ij}^{n+1})^2 < n(1+H^2)$. When $p \geq 3$, for fixed $\alpha \neq n+p$, we have

$$\sum_{\beta\neq n+p} N(H_\alpha H_\beta - H_\beta H_\alpha) = \sum_{ij\beta\neq\alpha,n+p}(h_{ij}^\beta)^2(h_{ii}^\alpha - h_{jj}^\alpha)^2.$$

Since

$$(h_{ii}^\alpha - h_{jj}^\alpha)^2 \leq 2[(h_{ii}^\alpha)^2 + (h_{jj}^\alpha)^2],$$

we get

$$\sum_{\beta\neq n+p} N(H_\alpha H_\beta - H_\beta H_\alpha) \leq 4\sum_{ij\beta\neq\alpha,n+p}(h_{ij}^\beta)^2(h_{ii}^\alpha)^2. \tag{2.18}$$

On the other hand, from (1.4) we obtain

$$\begin{aligned}R_{ii} &= (n-1) + nH^2 - H^2 - \sum_{j\beta\neq n+p}(h_{ij}^\beta)^2\\ &= (n-1)(1+H^2) - (h_{ii}^\alpha)^2 - \sum_{j\beta\neq\alpha,n+p}(h_{ij}^\beta)^2.\end{aligned} \tag{2.19}$$

Let Q be the infimum of the Ricci curvature of M^n. Then from (2.19) we get

$$\sum_{j\beta\neq\alpha,n+p}(h_{ij}^\beta)^2 \leq (n-1)(1+H^2) - (h_{ii}^\alpha)^2 - Q \tag{2.20}$$

and

$$\sum_{ij\beta\neq n+p}(h_{ij}^\beta)^2 \leq n(n-1)(1+H^2) - nQ. \tag{2.21}$$

The following inequality is obvious,

$$\sum_i (h_{ii}^\alpha)^4 \geq \frac{1}{n}\left[\sum_i (h_{ii}^\alpha)^2\right]^2 = \frac{1}{n}(\operatorname{tr} H_\alpha^2)^2. \tag{2.22}$$

Substituting (2.20) into (2.18) and using (2.22) we obtain

$$\begin{aligned}
&\sum_{\beta\neq n+p} N(H_\alpha H_\beta - H_\beta H_\alpha)\\
&\leq 4\sum_i [(n-1)(1+H^2) - (h_{ii}^\alpha)^2 - Q](h_{ii}^\alpha)^2\\
&= 4[(n-1)(1+H^2) - Q]\sum_i (h_{ii}^\alpha)^2 - 4\sum_i (h_{ii}^\alpha)^4\\
&\leq 4[(n-1)(1+H^2) - Q]\sum_i (h_{ii}^\alpha)^2 - \frac{4}{n}\sum_{\alpha\neq n+p}(\operatorname{tr} H_\alpha^2)^2,
\end{aligned}$$

and thus

$$\begin{aligned}
\sum_{\alpha,\beta\neq n+p} N(H_\alpha H_\beta - H_\beta H_\alpha) \leq &[(n-1)(1+H^2) - Q]\sum_{ij\alpha\neq n+p}(h_{ij}^\alpha)^2\\
&- \frac{4}{n}\sum_{\alpha\neq n+p}(\operatorname{tr} H_\alpha^2)^2.
\end{aligned} \tag{2.23}$$

Combining (2.21), (2.23) with (2.17) we obtain

$$\begin{aligned}
&\frac{1}{2}\Delta \sum_{ij\alpha\neq n+p}(h_{ij}^\alpha)^2\\
&\geq \sum_{ijk\alpha\neq n+p}(h_{ijk}^\alpha)^2 + [(-3n+4)(1+H^2) + 4Q]\sum_{ij\alpha\neq n+p}(h_{ij}^\alpha)^2\\
&\quad + \frac{4}{n}\sum_{\alpha\neq n+p}(\operatorname{tr} H_\alpha^2)^2 - \sum_{\alpha\neq n+p}(\operatorname{tr} H_\alpha^2)^2\\
&\geq \sum_{ijk\alpha\neq n+p}(h_{ijk}^\alpha)^2 + [(-3n+4)(1+H^2) + 4Q]\sum_{ij\alpha\neq n+p}(h_{ij}^\alpha)^2\\
&\quad - \frac{n-4}{n}\sum_{\alpha\neq n+p}(\operatorname{tr} H_\alpha^2)^2\\
&\geq \sum_{ijk\alpha\neq n+p}(h_{ijk}^\alpha)^2 + [(-3n+4)(1+H^2) + 4Q]\sum_{ij\alpha\neq n+p}(h_{ij}^\alpha)^2\\
&\quad - \frac{n-4}{n}[n(n-1)(1+H^2) - nQ]\sum_{ij\alpha\neq n+p}(h_{ij}^\alpha)^2\\
&= \sum_{ijk\alpha\neq n+p}(h_{ijk}^\alpha)^2 + [n(n-2)(1+H^2) + nQ]\sum_{ij\alpha\neq n+p}(h_{ij}^\alpha)^2.
\end{aligned} \tag{2.24}$$

Thus, we see from (2.24) that when $Q > (n-2)(1+H^2)$, then $\sum_{ij\alpha\neq n+p}(h_{ij}^{\alpha})^2 = 0$ and M^n is totally umbilical. This completes the proof of Theorem 3. ∎

References

[1] B. Y. Chen, *Some results of Chern–do Carmo–Kobayashi type and the length of the second fundamental form*, Indiana University Mathematics Journal **20** (1971), 1175–1185.

[2] S. S. Chern, M. do Carmo and S. Kobayashi, *Minimal submanifolds of a sphere with second fundamental form of constant length*, in *Functional Analysis and Related Fields*, Springer-Verlag, New York, 1970, pp. 60–75.

[3] N. Ejili, *Compact minimal submanifolds of a sphere with positive Ricci curvature*, Journal of the Mathematical Society of Japan **31** (1979), 251–256.

[4] A. M. Li and J. M. Li, *An intrinsic rigidity theorem for minimal submanifolds in a sphere*, Archiv der Mathematik **58** (1992), 582–594.

[5] H. Sun, *On spacelike submanifolds of a pseudoriemannian space form*, Note di Matematica **15** (1995), 215–224.

[6] H. Sun, *Pseudo-umbilical submanifolds of a space form* $N^{n+p}(C)$, Tsukuba Journal of Mathematics **20** (1996), 45–50.

[7] H. Sun, *Totally really pseudo-umbilical submanifolds of a complex projective space*, Tokyo Journal of Mathematics **20** (1997), 427–434.

[8] H. Sun, *Totally really pseudo-umbilical submanifolds of a quaternion space form*, Glasgow Mathematical Journal **40** (1998), 109–115.

[9] S. T. Yau, *Submanifolds with constant mean curvature II*, American Journal of Mathematics **97** (1975), 76–100.